Introduction to Perverse Sheaves, Part II

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1 Review

For this talk, $A = \Bbbk$ a field.

Theorem 1 (Global Verdier Duality)
Let $f : X \to Y$ be a continuous map. Then there is an additive triangulated functor $f^! : D^+(Y) \to D^+(X)$, called exceptional inverse image such that we have an adjunction

$$\mathrm{Hom}_{D^+(Y)}(Rf_!\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \mathrm{Hom}_{D^+(X)}(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet)$$

Definition 1.1. *Let $a_X : X \to pt$ be the usual projection to a point. Then the dualizing complex is defined as*

$$\omega_{X/\Bbbk} = a_X^!(\underline{\Bbbk}_{pt})$$

Definition 1.2 (Verdier Dual)**.** *Let X be a topological space and let $\mathcal{F}^\bullet \in D^b(X)$ define $\mathbb{D}_X(\mathcal{F}^\bullet) \in D^b(X)$*

$$\mathbb{D}_X(\mathcal{F}^\bullet) = R\mathscr{H}om^\bullet(\mathcal{F}^\bullet, \omega_X)$$

This will be a contravariant functor on $D^b(X)$.

Definition 1.3. *Let X be a topological space. A stratification of X is a partially ordered set $(\Lambda, \leq)$ and a collection of locally closed subsets $\{X_\lambda\}_{\lambda\in\Lambda}$ such that*

1. $X = \bigsqcup_{\lambda\in\Lambda} X_\lambda$ *and* $\overline{X_\lambda} = \bigsqcup_{\mu\leq\lambda} X_\mu$.
2. *Each X_λ is a smooth connected complex manifold.*

Definition 1.4. *A sheaf $\mathcal{F} \in \mathrm{mod}(\underline{\Bbbk}_X)$ is constructible if there exists a stratification $\bigsqcup_{\lambda\in\Lambda} X_\lambda$ such that $\mathcal{F}|_{X_\lambda}$ is a local system of finite rank for all $\lambda \in \Lambda$.*

Definition 1.5. *A complex $\mathcal{F}^\bullet \in D^b(X, \Bbbk)$ is constructible if all its cohomology sheaves*[1] *$\mathcal{H}^m\mathcal{F}^\bullet$ are constructible for some stratification Λ. Let*

$$D^b_c(X) := D^b_c(X, \Bbbk) = \Big\{\textit{full triangulated subcategory of } D^b(X) \textit{ consisting of constructible complexes}\Big\}$$

Theorem 1.6 (6 functors formalism)**.** *$D^b_c(X)$ is closed under the six operations*

$$Rf_*, Rf_!, f^{-1}, f^!, R\mathscr{H}om, \otimes^L$$

Corollary 1.7. *The dualizing sheaf ω_X is in $D^b_c(X)$. More generally $\mathbb{D}$ descends to a functor*

$$\mathbb{D} : D^b_c(X) \to D^b_c(X)$$

[1] If we were to ask that each term in $\mathcal{F}^\bullet$ is constructible, this would not be well defined in the derived category; a different representative might actually have different sheaves, as we only know that the cohomology sheaves are the same.

Theorem 2
Let $\mathcal{F}^\bullet \in D^b_c(X)$ where X is probably a complex analytic space. Then

(a) $\mathbb{D}^2 \cong \mathrm{id}$. *($\mathbb{D}$ is a duality.)*

(b) For $\mathcal{F}^\bullet \in D^b_c(X)$ we have $\mathbb{D}(\mathcal{F}^\bullet[n]) \cong \mathbb{D}(\mathcal{F}^\bullet)[-n]$.

(c) $\mathbb{D}\circ ! = * \circ \mathbb{D}$ (i.e. $\mathbb{D}\circ f_! = f_* \circ \mathbb{D}$) and vice versa.

(d) If X is smooth, and $\mathcal{L}$ is a local system, then $\mathbb{D}\mathcal{L} \cong \mathcal{L}^\vee[2\dim_{\mathbb{C}} X]$.

Proof. Almost every proof follows by some idea in key lemma of previous talk. Specifically, (a) follows by induction where the base case reduces to $D^b_c(pt) = D^b(\Bbbk - \mathrm{mod}^{fg})$. Here, we want to show that given a complex of $\Bbbk$ vector spaces, $M^\bullet$, the evaluation map

$$M^\bullet \to R\mathrm{Hom}^\bullet(R\mathrm{Hom}^\bullet(M^\bullet, \Bbbk), \Bbbk)$$

sending $m \mapsto \mathrm{ev}_m$ where $\mathrm{ev}_m(\varphi) = \varphi(m)$ where $\varphi \in R\mathrm{Hom}^\bullet(M^\bullet, \Bbbk)$ is an isomorphism. Now $\Bbbk$ is a field so we don't need to derive anything and for M a f.d. v.s, $M \cong (M^*)^*$. For the general case, we can use Noetherian induction by taking $i : Z \hookrightarrow X$ to be the inclusion of a closed subset to cut down the dimension and then use the distinguished triangles.

(b) is more or less by definition of a morphism of chain complexes. (c) follows from a sheafificiation of Global Verdier duality, aka local Verdier Duality

$$\mathbb{D}_Y(Rf_!\mathcal{F}^\bullet) = R\mathscr{H}om^\bullet(Rf_!\mathcal{F}^\bullet, \omega_Y) = Rf_*R\mathscr{H}om^\bullet(\mathcal{F}^\bullet, f^!\omega_Y) = Rf_*R\mathscr{H}om^\bullet(\mathcal{F}^\bullet, \omega_X)$$

and the RHS above is $Rf_*\mathbb{D}_X\mathcal{F}^\bullet$ as desired. (d) follows from a previously mentioned fact that for X smooth, of complex dimension $\dim_{\mathbb{C}} X$ (and thus real dimension $2\dim_{\mathbb{C}} X$), and $\mathcal{L}$ a local system,

$$\mathbb{D}(\mathcal{L}) = \mathcal{L}^\vee \otimes \mathcal{L}_{or}[2\dim_{\mathbb{C}} X]$$

as X being a complex analytic space is orientable and thus $\mathcal{L}_{or} = \underline{A}_X$.

Corollary 1.8. *If X is smooth, then $\mathbb{C}_X[\dim_{\mathbb{C}} X]$ is self dual under $\mathbb{D}$.*

2 Perverse Sheaves

We are finally now in a position to define a perverse sheaf. If $\mathcal{F}$ is a constructible sheaf under the stratification Λ, as each X_λ is connected, the dimension of the vector space $\mathcal{F}_x$ is the same for any $x \in X_\lambda$ since X_λ is connected. Thus to a constructible sheaf we can make a table

λ	$\mathbb{C}^{n_\lambda}$
μ	$\mathbb{C}^{n_\mu}$
ν	$\mathbb{C}^{n_\nu}$

where the entry in row λ is the vector space $\mathcal{F}_x$ where x is any point in X_λ. Now let $\mathcal{F}^\bullet$ be a constructible complex so that each cohomology sheaf $\mathcal{H}^i\mathcal{F}^\bullet$ is constructible, we have more columns corresponding to different i, the cohomological degree. The corresponding vector space at the stratum λ is denoted $h^i(\mathcal{F}^\bullet_\lambda)$ and this data is also collected into a table called the table of stalks as seen below.

	...	-1	0	1	...
λ			$h^0(\mathcal{F}^\bullet)_\lambda$		
μ				$h^1(\mathcal{F}^\bullet)_\mu$	
ν					

Example 1. The constant sheaf $\mathbb{C}_X$ on X has the following table.

	0	1
λ	$\mathbb{C}$	0
μ	$\mathbb{C}$	0
ν	$\mathbb{C}$	0

Definition 2.1 ((middle-perversity) perverse sheaf)**.** *Let $d_\lambda = \dim_{\mathbb{C}} X_\lambda$. A complex of sheaves $\mathcal{F}^\bullet \in D_c^b(X)$ is called a perverse sheaf if the following conditions are satisfied.*

1. *The table of stalks for $\mathcal{F}^\bullet$ has the following form*

	$-d_\lambda$			$-d_\mu$		$-d_\nu$	
λ	$*$	0	0	0	0	0	0
μ	$*$	$*$	$*$	$*$	0	0	0
ν	$*$	$*$	$*$	$*$	$*$	$*$	0

2. *The same is true for $\mathbb{D}(\mathcal{F}^\bullet)$.*

Aka the table of stalks for $\mathcal{F}^\bullet$ should be lower triangular, except the diagonal is a bit jagged.

Example 2. $\mathbb{C}_X[\dim_{\mathbb{C}} X]$ is a perverse sheaf. It is self dual so we only need to check one condition and as $\dim_{\mathbb{C}} X > \dim_{\mathbb{C}} X_\lambda$ for any λ we are definitely lower triangular.

	$-\dim_{\mathbb{C}} X$	...
X_λ	$\Bbbk$	0
X_μ	$\Bbbk$	0

Example 3. Consider the stratification of the flag variety for $\mathfrak{sl}_2$, aka $\mathbb{P}^1$ given by the Schubert decomposition $\mathbb{P}^1 = \mathbb{A}^1 \sqcup \{pt\}$. Let $j : \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$, is $j_!(\Bbbk_{\mathbb{A}^1}[1])$ perverse?

For a locally closed inclusion $j_!$ is always extension by zero. Because extension by zero is exact, we don't need to derive $j_!$ in this case and thus we see that the table of stalks for $j_!(\Bbbk_{\mathbb{A}^1}[1])$ is

	-1	0
$\mathbb{A}^1$	$\Bbbk$	0
pt	0	0

Now note that $\mathbb{D}(j_!(\Bbbk_{\mathbb{A}^1}[1])) = j_*(\mathbb{D}(\Bbbk_{\mathbb{A}^1}[1])) = j_*(\Bbbk_{\mathbb{A}^1}[1])$. In order to compute the table of stalks for $j_*(\Bbbk_{\mathbb{A}^1}[1])$ you need to use one of the adjunction triangles, TR2 of a triangulated category, +more stuff to obtain the distinguished triangle

$$\Bbbk_{\mathbb{P}^1}[1] \to Rj_*(\Bbbk_{\mathbb{A}^1}[1]) \to i_*\Bbbk_{pt}$$

where $i : pt \hookrightarrow \mathbb{P}^1$. i is closed and therefore i_* is extension by zero. Because distinguished triangles in $D_c^b(\mathbb{P}^1)$ induce LES on hypercohomology, it follows that the table of stalks for $Rj_*(\Bbbk_{\mathbb{A}^1}[1])$ will be

	-1	0
$\mathbb{A}^1$	$\Bbbk$	0
pt	$\Bbbk$	$\Bbbk$

It follows that both $j_!(\Bbbk_{\mathbb{A}^1}[1])$ and $j_*(\Bbbk_{\mathbb{A}^1}[1])$ are perverse.

Remark. The category of Perverse sheaves $\text{Perv}(X) \subset D_c^b(X)$ is abelian. This is why perverse sheaves show up so often in geometric representation theory.

Theorem 2.2. *Simple objects in* $\text{Perv}(X)$ *are called intersection cohomology sheaves. If all the strata are simply connected, then the IC sheaves are in bijection with the poset* Λ *and will be denoted* IC_λ *for* $\lambda \in \Lambda$*. Each* IC_λ *is uniquely specified by the following two conditions.*

1. IC_λ *is Verdier self dual, and*
2. *The table of stalks for* IC_λ *is of the form*

		$-d_\lambda$			$-d_\mu$		$-d_\nu$	
$\not< \lambda$	0	0	0	0	0	0	0	0
λ	0	$\Bbbk$	0	0	0	0	0	0
μ	0	$*$	$*$	$*$	0	0	0	0
ν	0	$*$	$*$	$*$	$*$	$*$	0	0

I.E. the only nonzero term appearing on the diagonal is on the λ *spot.*

Example 4. In our example above, what are the IC of $\mathbb{P}^1 = \mathbb{A} \sqcup \{pt\}$? We claim $\text{IC}_{\mathbb{A}^1} = \Bbbk_{\mathbb{P}^1}[1]$ and $\text{IC}_{pt} = i_*(\Bbbk_{pt})$. The table of stalks for $\Bbbk_{\mathbb{P}^1}$ is

	-1	0
$\mathbb{C}^\times$	$\Bbbk$	0
pt	$\Bbbk$	0

Moreover it is self dual as $\dim_{\mathbb{C}} \mathbb{P}^1 = 1$. Therefore it satisfies the condition to be $\text{IC}_{\mathbb{A}^1}$. As i is a closed inclusion $i_* = i_!$ is extension by zero and therefore the table of stalks of $i_*(\Bbbk_{pt})$ is

	-1	0
$\mathbb{C}^\times$	0	0
pt	0	$\Bbbk$

We now check that $\mathbb{D}(i_*(\Bbbk_{pt})) = i_!\mathbb{D}(\Bbbk_{pt}) = i_*\Bbbk_{pt}$ so it is self dual and thus satisfies the condition to be IC_{pt}.

Remark. For type A, we have an alternative description of the BS variety $Y(\underline{w})$ where $\underline{w} = s_{\alpha_1} \dots s_{\alpha_d}$ as follows. First let $s_i = (i\ i+1)$. Then a point in $y \in Y(\underline{w})$ is a sequence of $d+1$ complete flags (aka $Y(\underline{w}) \subset (G/B)^{d+1}$)

$$y = (\text{std}, \mathcal{F}_1^\bullet, \dots, , \mathcal{F}_d^\bullet)$$

defined inductively as follows. The first is always the standard flag and $V_i^\bullet = V_{i+1}^\bullet$ <u>except</u> at the α_i spot where $V_i^0 = 0$, V_i^1 is a line, etc. For example, if $G = \text{GL}_2$ and $\underline{w} = s_1 s_1$, then the conditions on $y = (\text{std}, \mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet) \in Y(\underline{w})$ where

$$\text{std} = 0 \subset \{e_1\} \subset \mathbb{C}^2$$
$$\mathcal{F}_1^\bullet = 0 \subset V_1^1 \subset \mathbb{C}^2$$
$$\mathcal{F}_2^\bullet = 0 \subset V_2^1 \subset \mathbb{C}^2$$

are given by[2]

$$\begin{aligned} s_1 \text{ condition} : 0 \subset V_1^1 \subset \mathbb{C}^2 & \qquad \text{and } V_1^1 = \{e_1\} \\ s_1 \text{ condition} : 0 \subset V_2^1 \subset \mathbb{C}^2 & \qquad \text{and } V_2^2 = V_1^2 \end{aligned}$$

The multiplication map $\overline{\text{mult}} : Y(\underline{w}) \to G/B$ is now just

$$\overline{\text{mult}}(\text{std}, \mathcal{F}_1^\bullet, \ldots, , \mathcal{F}_d^\bullet) = \mathcal{F}_d^\bullet$$

$\overline{\text{mult}}$ is always proper and when $\underline{w}$ is a reduced expression then one can show that the image is exactly $\overline{BwB/B}$. Furthermore $Y(\underline{w})$ being an iterated $\mathbb{P}^1$ bundle is smooth (smoothness is a local property) and in fact $Y(\underline{w}) \to \overline{BwB/B}$ is a resolution of singularities. Note that $\dim_{\mathbb{C}} Y(\underline{w}) = |\underline{w}|$, the number of terms in the expression $\underline{w}$, essentially because each s_i gives you one more degree of freedom.

Definition 2.3. *The action of $w \in S_n$ on the standard flag* $\text{std} = 0 \subset \{e_1\} \subset \{e_1, e_2\} \subset \ldots \subset \mathbb{C}^n$ *in* GL_n *is given by*

$$w \cdot \text{std} = 0 \subset \left\{e_{w(1)}\right\} \subset \left\{e_{w(1)}, e_{w(2)}\right\} \subset \ldots \subset \mathbb{C}^n$$

Theorem 2.4 (Proper Base Change)**.** *Let $f : X \to Y$ be a proper map of locally compact spaces. If $\mathcal{F} \in \text{Sh}(X)$, there is a functorial isomorphism*

$$\left(R^k f_* \mathcal{F}\right)_y \cong H^k(X_y, \mathcal{F}_y) \quad \forall y \in Y$$

Example 5. Consider $\overline{\text{mult}} : Y(s, s) \to \text{GL}_2/B = \mathbb{P}^1$. What is the table of stalks for $\overline{\text{mult}}_*(\Bbbk_Y[2])$? By the above theorem we see it suffices to compute cohomology of the fibers over the two stratum of $\mathbb{P}^1$.

For the fiber over the *pt* aka the fiber over the standard flag std we see that the fiber over id will be the set of all tuples

$$(\text{std}, \mathcal{F}_1^\bullet, \text{std})$$

where

$$\begin{aligned} \text{std} &= 0 \subset \{e_1\} \subset \mathbb{C}^2 \\ \mathcal{F}_1^\bullet &= 0 \subset V_1^1 \subset \mathbb{C}^2 \\ \text{std} &= 0 \subset \{e_1\} \subset \mathbb{C}^2 \end{aligned}$$

satisfying the conditions

- $0 \subset V_1^1 \subset \mathbb{C}^2$
- $0 \subset \{e_1\} \subset \mathbb{C}^2$.

It's clear that only the first condition contributes and that the set of all such lines in $\mathbb{C}^2$ is exactly $\mathbb{P}^1\mathbb{C}$. Thus by proper base change, it follows that

$$\mathcal{H}^* \overline{\text{mult}}_*(\Bbbk_Y[2])_{std} = H^*(\mathbb{P}^1, \Bbbk_{\mathbb{P}^1}[2])$$

When we do the computation over the fiber over $s\text{std} \in \mathbb{P}^1$, corresponding to $s = (12)$, the only thing that changes is that the last flag is now $0 \subset \{e_2\} \subset \mathbb{C}^2$ so we again have that the fiber is $\mathbb{P}^1_{\mathbb{C}}$. Accounting for the [2] shift it follows that the table of stalks for $\overline{\text{mult}}_*(\Bbbk_Y[2])$ is exactly

[2] The conditions are combinations of equalities and Gelfand-Tseltlin patterns.

	-2	0
$\mathbb{A}^1 = X_s$	$\Bbbk$	$\Bbbk$
pt	$\Bbbk$	$\Bbbk$

But this is exactly the table of stalks for $\mathrm{IC}_{\mathbb{A}^1}[1] \oplus \mathrm{IC}_{\mathbb{A}^1}[-1]$! Wouldn't it be great if they were isomorphic?

Definition 2.5. *A complex is called a semisimple complex if it is isomorphic to a complex of the form* $\bigoplus \mathrm{IC}_\lambda[k]^{\oplus m_{\lambda,k}}$ *where* $m_{\lambda,k} \in \mathbb{N}$

Theorem 3 (Decomposition Theorem)
If $f : Y \to X$ is a proper morphism of algebraic varieties where Y is smooth, then f_* applied to a semisimple complex is a semisimple complex.

With this we can now show $\overline{\mathrm{mult}}_*(\Bbbk_{Y(s,s)}[2]) \cong \mathrm{IC}_{\mathbb{A}^1}[1] \oplus \mathrm{IC}_{\mathbb{A}^1}[-1]$ using a combinatorial algorithm as follows

Algorithm for decomposing semsimple complexes:

0. Assume you know the table of stalks for $\mathrm{IC}_\lambda, \forall \lambda \in \Lambda$.
1. If $\mathcal{F}^\bullet$ is semisimple, look at the row whose stratum λ_0 is indexed by the longest element (in general, a maximal element in poset) and suppose we have a nonzero entry, say at homological degree $-d_{\lambda_0} + k$ where $k \in \mathbb{Z}$. Because $\mathcal{F}^\bullet$ is semisimple, it's a direct sum of IC_λ possibly with shifts. But because IC_λ can only contribute to row μ if $\mu \leq \lambda$, we see that no other IC_λ can contribute to the table of stalks in row λ_0 other than IC_{λ_0}.
2. It follows from above that $\mathrm{IC}_{\lambda_0}[-k]$ must appear as a direct summand of $\mathcal{F}^\bullet$. Delete the table of stalks of $\mathrm{IC}_{\lambda_0}[-k]$ from $\mathcal{F}^\bullet$ and repeat until we only have 0's in row λ_0.
3. Repeat the procedure above with a stratum this is maximal with respect to the subposet $\Lambda \setminus \{\lambda_0\}$. Repeat until we covered the entire table.

As $Y = Y(s,s)$ is smooth, and therefore is stratifed trivially by say e and we have that $\Bbbk_Y[\dim_{\mathbb{C}} Y] = \Bbbk_Y[2] = \mathrm{IC}_e$ is a semisimple complex and thus by the decomposition theorem $\overline{\mathrm{mult}}_*(\Bbbk_Y[2])$ is a semisimple complex and we can apply our algorithm which easily shows the desired isomorphism. Another way to rephrase the algorithm is that

Theorem 2.6. *Semisimple complexes are determined up to isomorphism by their table of stalks.*

3 Proof of the KL Conjectures

Example 6. Recall $b_s = \delta_s + v$ and that $(\delta_s + v)(\delta_s - v^{-1}) = 0$. Compute b_s^2 to see that $b_s^2 = (v + v^{-1})b_s$.

Definition 3.1. *Given a complex $\mathcal{F}^\bullet \in D^b_c(G/B)$, define a map $\mathrm{ch} : D^b_c(G/B) \to \mathcal{H}(W)$ via*

$$\mathrm{ch}(\mathcal{F}^\bullet) = \sum_{w \in W} \left(\sum_{i \in \mathbb{Z}} \dim h^{-d_w - i} \mathcal{F}^\bullet_{x_w} v^i \right) \delta_w, \quad x_w \text{ any point in } X_w$$

The example above will then motivate

Theorem 3.2.

$$\operatorname{ch}(\overline{\operatorname{mult}}_*(\Bbbk_{Y(s_1,\dots,s_d)}[d])) = b_{s_1}\dots b_{s_d}$$

In fact we can say more,

Theorem 4 *(a) Let $\mathcal{F}^\bullet$ be a semisimple complex in $D^b_c(G/B)$. Then $h\left(\mathbb{D}(\mathcal{F}^\bullet)\right) = \overline{h(\mathcal{F}^\bullet)}$.*

(b) Let $X = G/B$ with the Bruhat stratification. Then $h\left(\mathrm{IC}_w\right) = b_w$.

Proof. (a) Akin to our calculations before, one can show using the decomposition theorem that when $(s_1,\dots,s_d)$ is a reduced expression we have

$$\overline{\operatorname{mult}}_*(\Bbbk_{Y(s_1,\dots,s_d)}[d]) \cong \mathrm{IC}_{s_i\dots s_d} \bigoplus_{\mu<s_1\dots s_d} \mathrm{IC}_\mu[k]^{m_{\mu,k}}$$

(This is because the fiber over any point in $X_{s_i\dots s_d}$ is a point because of the resolution of singularities) Now let $\mathcal{F}^\bullet = \oplus \mathrm{IC}_\lambda[k]^{\oplus m_{\lambda,k}}$ be a semisimple complex. Since ch is additive, it follows that $\operatorname{ch}(\mathcal{F}^\bullet)$ is a linear sum of elements of the form $\operatorname{ch}(\overline{\operatorname{mult}}_*(\Bbbk_{Y(s_1,\dots,s_d)}[d])[k])$ by repeated applications of the isomorphism above noting we get smaller in Bruhat order with each application and thus we terminate. It therefore suffices to prove in this case. But we have

$$\operatorname{ch}\left(\mathbb{D}(\overline{\operatorname{mult}}_*(\Bbbk_{Y(s_1,\dots,s_d)})[d])\right) = \operatorname{ch}\left(\overline{\operatorname{mult}}_*(\mathbb{D}(\Bbbk_{Y(s_1,\dots,s_d)}[d]))\right) = \operatorname{ch}\left(\overline{\operatorname{mult}}_*(\Bbbk_{Y(s_1,\dots,s_d)}[d])\right) = b_{s_1}\dots b_{s_d}$$

Likewise we see that

$$\overline{\operatorname{ch}(\overline{\operatorname{mult}}_*(\Bbbk_{Y(s_1,\dots,s_d)})[d]))} = \overline{b_{s_1}\dots b_{s_d}} = b_{s_1}\dots b_{s_d}$$

And given a complex $\mathcal{G}^\bullet$ satisfying (a) we have that

$$\operatorname{ch}(\mathbb{D}(\mathcal{G}^\bullet[k])) = \operatorname{ch}(\mathbb{D}(\mathcal{G}^\bullet)[-k]) = v^k h(\mathbb{D}(\mathcal{G}^\bullet)) = v^k\overline{h(\mathcal{G}^\bullet)} = \overline{v^{-k}h(\mathcal{G}^\bullet)} = \overline{h(\mathcal{G}^\bullet[k])}$$

(b) Because IC_w is Verdier self dual, using (a) we see that

$$\overline{\operatorname{ch}\left(\mathrm{IC}_w\right)} = \operatorname{ch}\left(\mathbb{D}(\mathrm{IC}_w)\right) = \operatorname{ch}\left(\mathrm{IC}_w\right)$$

Thus $\{\operatorname{ch}\left(\mathrm{IC}_w\right)\}$ satisfies condition 1 of being a KL basis. It therefore suffices to show $\operatorname{ch}\left(\mathrm{IC}_w\right)$ satisfies the degree bound. By definition of IC_w the only nonzero entry in row w is at $-d_w$ and it's of dimension 1 so we see that

$$\operatorname{ch}\left(\mathrm{IC}_w\right) = \delta_w + \sum_{\mu<w}\left(\sum_{i\in\mathbb{Z}} \dim h^{-d_\mu - i}(\mathrm{IC}_w)_{x_\mu} v^i\right)\delta_\mu$$

But by definition, the μ row of IC_λ will be zero starting at $-d_\mu$ and going to the right when $\mu \neq \lambda$. This means that

$$\left(\sum_{i\in\mathbb{Z}} \dim h^{-d_\mu - i}(\mathrm{IC}_w)_{x_\mu} v^i\right) \in v\mathbb{Z}[v]$$

so $\operatorname{ch}\left(\mathrm{IC}_w\right)$ satisfies the second condition to be a KL basis. Since a KL basis is unique we are done.

A direct consequence of part (b) is that

$$h_{\mu,w}(v) = \sum_{i\in\mathbb{Z}} \dim h^{-d_\mu - i}(\mathrm{IC}_w)_{x_\mu} v^i \tag{1}$$

Part (b) also gives more insight into what happened last week. Specifically, after applying BB localization and RH, we land in $\text{Perv}^{\text{B}}(G/B)$. Clearly character formulas just depend on $K_0(\text{Perv}^{\text{B}}(G/B))$ and the claim is there is an isomorphism given by

$$\chi : K_0(\text{Perv}^{\text{B}}(G/B)) \xrightarrow{\sim} \mathbb{Z}[W]$$
$$\chi([M^\bullet]) = \sum_{w\in W}\sum_{i\in\mathbb{Z}}(-1)^i h^i(M^\bullet_{x_w})[w]$$

which one can see by noting that $[\mathcal{M}_w] = [(j_w)_!(\mathbb{C}_{X_w}[\ell(w)])]$(aka the Verma modules) form a basis for $K_0(\text{Perv}^{\text{B}}(G/B))$ and note that

$$\chi([\mathcal{M}_w]) = (-1)^{\ell(w)}[w] \tag{2}$$

is sent to a basis for $\mathbb{Z}[W]$. But now applying χ to IC_w we see that

$$\begin{aligned}\chi([\text{IC}_w]) &= \sum_{g\in W}\sum_{i\in\mathbb{Z}}(-1)^i h^i((\text{IC}_w)_{x_g})[g] \\ &\overset{Eq.~(1)}{=\!=} \sum_{g\in W}(-1)^{d_g}h_{g,w}(-1)[g] \\ &\overset{Eq.~(2)}{=\!=} \sum_{g\in W}(-1)^{\ell(g)-\ell(g)}h_{g,w}(-1)\chi([\mathcal{M}_g]) \\ &= \sum_{g\in W}h_{g,w}(-1)\chi([\mathcal{M}_g])\end{aligned}$$

where the second equality comes from Eq. (1) and noting

$$v^{d_g}h_{g,w}(v) = \sum_{i\in\mathbb{Z}}\dim h^{-d_g-i}(\text{IC}_w)_{x_g}v^{i+d_g} = \sum_{i\in\mathbb{Z}}\dim h^i(\text{IC}_w)_{x_g}v^{-i}$$

and plugging in $v=-1$. Because χ is an isomorphism this means back in $K_0(\text{Perv}^{\text{B}}(G/B)$ we have the formula

$$[\text{IC}_w] = \sum_{g\in W}h_{g,w}(-1)[\mathcal{M}_g]$$

Finally, Exercise 3.28 in [EMTW], Springer Version states that

$$v^{-(\ell(w)-\ell(g))}h_{g,w}(v) \in \mathbb{Z}[v^{-2}]$$

Thus plugging in $v=-1$ is the same as plugging in $v=1$ above so we see that

$$h_{g,w}(-1) = (-1)^{\ell(w)-\ell(g)}h_{g,w}(1)$$

And modulo $P_{g,w} = v^{something}h_{g,w}$ this should be the final form of the KL conjectures you see on wikipedia.

$$\boxed{\text{ch}(L_w) = \sum_{g\le w}(-1)^{\ell(w)-\ell(g)}h_{g,w}(1)\text{ch}(M_y)}$$

Part (b) suggests that Theorem 3.2 is a decategorification of a richer structure. Namely it suggests that we should we have an operation $\star$ called convolution on (semisimple) perverse sheaves such that

$$\overline{\text{mult}}_*(\Bbbk_{Y(s_1,\ldots,s_d)}[d]) \cong \text{IC}_{s_1}\star\ldots\star\text{IC}_{s_d}$$

and then Theorem 3.2 will follow by applying h. It turns out that there are objects called $B-$equivariant perverse sheaves $D^b_B(G/B,\Bbbk)$ (which are slightly different than the category we have been working with, $B-$ constructible sheaves on G/B) and we do have a monoidal structure given by convolution $\star$ on this category. In other words,

Theorem 5 (Geometric Hecke Category)
The geometric Hecke category $\mathcal{H}^{geo}$ is defined to be the monoidal category

$$\mathcal{H}^{geo} := \langle \mathrm{IC}_w \,| w \in W\rangle_{\star,[1],\oplus} \subset D^b_B(G/B)$$

i.e. smallest subcategory containing $\{\mathrm{IC}_w\}$ closed under convolution, homological shifts, and direct summands. We then have that there is an isomorphism of algebras

$$K_0\left(\mathcal{H}^{geo}\right)_{\oplus} \cong \mathcal{H}(W)$$

4 Addendum

Set $\mathscr{B} = G/B$ and $\mathscr{P}^s = G/P_s$ the partial flag variety associated to the maximal parabolic subgroup P_s. Note that

$$W^s = \{w \in W | \ell(ws) > \ell(w)\}$$

is a set of representatives for $W/\{e, s\}$ and the Bruhat decomposition provides a decomposition

$$\mathscr{P}^s = \bigsqcup_{w\in W^s} \mathscr{P}^s_w \qquad \mathscr{P}^s_w = BwP_s/P_s \simeq \mathbb{A}^{\ell(w)}_{\mathbb{C}}$$

Let $\pi_s : \mathscr{B} \to \mathscr{P}^s$ and for $s_1, \ldots, s_n \in S$, set

$$\mathcal{E}(s_1, \ldots, s_n) = (\pi_{s_n})^{-1}(R\pi_{s_n})_* \ldots (\pi_{s_1})^{-1}(R\pi_{s_1})_*(\Bbbk_{G/B}[n]) \in D^b_{(B)}(\mathscr{B}, \Bbbk)$$

Proposition 4.1. *Let $\mathcal{F}^\bullet \in D^b_{(B)}(\mathscr{B}, \Bbbk)$ such that $\mathbb{H}^k(\mathcal{F}^\bullet) = 0$ unless k is even and let $s \in S$. Then $\mathbb{H}^k((\pi_s)^{-1}(R\pi_s)_*(\mathcal{F}^\bullet)) = 0$ unless k is even and*

$$\mathrm{ch}((\pi_s)^{-1}(R\pi_s)_*(\mathcal{F}^\bullet)) = \mathrm{ch}(\mathcal{F}^\bullet)v^{-1}b_s \tag{3}$$

Proof. For any $y \in W$ se have

$$\mathbb{H}^k(\,(\pi_s^{-1}(R\pi_s)_*(\mathcal{F}^\bullet)_{yB}) = \mathbb{H}^k(\,(R\pi_s)_*(\mathcal{F}^\bullet)_{yP_s}) = \mathcal{H}^k(\,(R\pi_s)_*(\mathcal{F}^\bullet))_{yP_s} = \mathbb{H}^k(\pi_s^{-1}(yP_s), \mathcal{F}^\bullet|_{\pi_s^{-1}(yP_s)})$$

where we have used that taking stalks is exact and derived proper base change(c.f. Sheaves in Topology Notes). Before proceeding, we record a lemma for use.

Lemma 4.2 (Springer). *For $s \in W$ a simple reflection and any $w \in W$, we have*

$$(BwB)(BsB) = \begin{cases} BwsB & \ell(ws) > \ell(w) \\ BwsB \cup BwB & \ell(ws) < \ell(w) \end{cases}$$

First case: $ys > y$. We have that

$$\pi_s^{-1}(yP_s) = \{ygB | g \in P_s\} \simeq P_s/B \simeq \mathbb{P}^1_{\mathbb{C}} \qquad ygB \mapsto gB$$

and since $ys > y$, Lemma 4.2 will show that for $g \in P_s$

$$ygB \in \begin{cases} BysB/B & \text{if } g \notin B \\ ByB/B & \text{if } g \in B \end{cases}$$

Indeed, as $P_s = B \sqcup BsB$, in the first case above we see that $g \in BsB$. But under the isomorphism $\pi_s^{-1}(yP_s) \simeq P_s/B \simeq \mathbb{P}^1_{\mathbb{C}}$ we see that

$$\mathbb{A}^1_{\mathbb{C}} \longleftrightarrow \{ygB | g \in P_s,\ g \notin B\} \subseteq BysB/B$$
$$\text{pt} \longleftrightarrow \{ygB | g \in P_s,\ g \in B\} \subseteq ByB/B$$

Decomposing $\mathbb{P}^1_{\mathbb{C}} = \mathbb{A}^1_{\mathbb{C}} \sqcup \text{pt}$, it follows that we have the LES

$$\ldots \to \mathbb{H}^k_c(\mathbb{A}^1_{\mathbb{C}}, \mathcal{F}^\bullet|_{\mathbb{A}^1}) \to \mathbb{H}^k(\,(\pi_s^{-1}(R\pi_s)_*(\mathcal{F}^\bullet)_{yB}) \to \mathbb{H}^k(\text{pt}, \mathcal{F}^\bullet|_{\text{pt}}) \to \ldots \tag{4}$$

from [Sheaves in Topology Notes, Section 2] as $\mathbb{P}^1_{\mathbb{C}}$ and pt are compact. From above we see that $\mathbb{A}^1_{\mathbb{C}} \subseteq BysB/B$ and so $\mathcal{H}^k\mathcal{F}^\bullet|_{\mathbb{A}^1}$ is a local system, and likewise $\mathcal{H}^k\mathcal{F}^\bullet|_{\text{pt}}$ is a local system. But $\mathbb{A}^1_{\mathbb{C}}, \text{pt}$ are connected, simply connected and therefore $\mathcal{H}^k\mathcal{F}^\bullet|_{\mathbb{A}^1}, \mathcal{H}^k\mathcal{F}^\bullet|_{\text{pt}}$ are in fact constant sheaves corresponding to $\mathcal{H}^k\mathcal{F}^\bullet_{ysB}$ and $\mathcal{H}^k\mathcal{F}^\bullet_{yB}$ respectively.

Now, suppose we have an injective resolution of $\mathcal{F}^k$ for each k, then we can construct a Cartan-Eilenberg resolution $\mathcal{I}^{\bullet,\bullet}$ whose total complex gives us a resolution $\mathcal{F}^\bullet \to \text{Tot}(\mathcal{I}^{\bullet,\bullet})$. Using the spectral sequence for double complexes and taking cohomology horizontally first, we obtain

$$H^p_c(\mathbb{A}^1_{\mathbb{C}}, \mathcal{H}^q(\mathcal{F}^\bullet|_{\mathbb{A}^1_{\mathbb{C}}})) \implies H^{p+q}_c(\mathbb{A}^1_{\mathbb{C}}, \mathcal{F}^\bullet|_{\mathbb{A}^1})$$

Alternatively the spectral sequence above comes from the Leray spectral sequence for cohomology with compact support with $f = \text{id}$. Now because $\mathcal{H}^q(\mathcal{F}^\bullet|_{\mathbb{A}^1_{\mathbb{C}}}) = \underline{\mathbb{k}}^m$ and

$$H^j_c(\mathbb{A}^1_{\mathbb{C}}, \underline{\mathbb{k}}) = \begin{cases} \mathbb{k} & \text{if } j = 2 \\ 0 & \text{otherwise} \end{cases}$$

it follows that

$$H^{q-2}(\mathcal{F}^\bullet_{ysB}) \cong H^q_c(\mathbb{A}^1_{\mathbb{C}}, \mathcal{F}^\bullet|_{\mathbb{A}^1})$$

We similarly have $\mathbb{H}^k(\text{pt}, \mathcal{F}^\bullet|_{\text{pt}}) = H^k(\mathcal{F}^\bullet_{yB})$ and therefore the LES in Eq. (4) becomes

$$\ldots \to H^{k-2}(\mathcal{F}^\bullet_{ysB}) \to \mathbb{H}^k(\,(\pi_s^{-1}(R\pi_s)_*(\mathcal{F}^\bullet)_{yB}) \to H^k(\mathcal{F}^\bullet_{yB}) \to \ldots$$

By assumption $\mathbb{H}^k(\mathcal{F}^\bullet) = 0$ unless k is even and therefore the LES breaks up into SES such that

$$\dim_{\mathbb{k}} \mathbb{H}^k(\,(\pi_s^{-1}(R\pi_s)_*(\mathcal{F}^\bullet)_{yB}) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \dim_{\mathbb{k}} H^{k-2}(\mathcal{F}^\bullet_{ysB}) + \dim_{\mathbb{k}} H^k(\mathcal{F}^\bullet_{yB}) & \text{if } k \text{ is even} \end{cases}$$

Second case: $ys < y$ One can repeat the arguments above to obtain

$$\dim_{\mathbb{k}} \mathbb{H}^k(\,(\pi_s^{-1}(R\pi_s)_*(\mathcal{F}^\bullet)_{yB}) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \dim_{\mathbb{k}} H^{k-2}(\mathcal{F}^\bullet_{yB}) + \dim_{\mathbb{k}} H^k(\mathcal{F}^\bullet_{ysB}) & \text{if } k \text{ is even} \end{cases}$$

These two cases will then correspond to

$$T_w(v^{-1}b_s) = \begin{cases} v^{-1}T_{ws} + T_w & \text{if } ws > w \\ v^{-2}T_{ws} + v^{-1}T_w & \text{if } ws < w \end{cases}$$

on the RHS of Eq. (3) after expanding out $\text{ch}(\mathcal{F}^\bullet)v^{-1}b_s$.

Corollary 6

For any $s_1, \ldots, s_n \in S$ *and* $w \in W$*, we have*

$$\mathbb{H}^i(\mathcal{E}(s_1, \ldots, s_n)_{w\mathscr{B}}) = 0 \text{ unless } i \equiv n \pmod 2$$

Moreover we have

$$\mathrm{ch}(\mathcal{E}(s_1, \ldots, s_n)) = b_{s_1} \ldots b_{s_n}$$